

# On the stability of plane Poiseuille flow to finite-amplitude disturbances, considering the higher-order Landau coefficients

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In this work a study has been made of the Stuart (1960)–Watson (1960) formalism as applied to plane Poiseuille flow. In particular, the higher-order Landau coefficients have been calculated for the Reynolds & Potter (1967) method and for the Watson (1960) method. The results have been used to study the convergence of the Stuart–Landau series. A convergence curve in the  $(\alpha, R)$ -plane has been obtained by using suitable Domb–Sykes plots. In the region of poor convergence of the series, and also in a part of the divergent region of the series, it has been found that the Shanks (1955) method, using the  $e_1^m$  transformation, serves as a very effective way of finding the proper sum of the series, or of finding the proper antilimit of the series. The results for the velocity calculations at  $R = 5000$  are in very good agreement with Herbert's (1977) Fourier-truncation method using  $N = 4$ . The Watson method and the Reynolds & Potter method have also been compared in the subcritical and supercritical regions. It is found in the supercritical region that there is not much difference in the results by the 'true problem' of Watson and the 'false problem' of Reynolds & Potter when the respective series in both methods are summed by the Shanks method. This fact could possibly be capitalized upon in the subcritical region, where the Watson method is difficult to apply.

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## 1. Introduction

The study of the stability of plane Poiseuille flow using a Stuart–Landau-type series to describe the growth rate of the disturbance amplitude, has been of considerable interest in the past. Pioneering works in this field, following Landau (1944), are those of Stuart (1960) and Watson (1960), followed by those of Eckhaus (1965), Reynolds & Potter (1967) and Itoh (1974*a, b*, 1977*a, b*). Close scrutiny of these works, starting with Stuart's, reveals that despite many similarities there are also important points of differences. Some of these differences are obvious, but some are subtle though important. For instance, Eckhaus uses the method of eigenfunction expansions to solve his equations, and Itoh uses Eckhaus's mathematics but develops the problem mainly according to his own philosophy.

In the present work a study has been made of the Reynolds & Potter (1967) method (hereinafter to be referred to as RP) in the subcritical region, to obtain the region of convergence of the Stuart–Landau series, and to calculate the equilibrium amplitudes and velocity distributions. It is found that, by using the  $e_1^m$  transformation

of Shanks (1955), the RP method can be used effectively not only in the region of poor convergence of the Stuart–Landau series, but also in certain portions of the divergent region. Further, a comparison of the ‘false problem’ of RP and the ‘true problem’ of Watson has been made. In the supercritical region it is found by using the Shanks method that the ‘false problem’ and the ‘true problem’ give substantially the same sums for the series at different levels of amplitude. It is believed that, from the practical viewpoint, the results will have relevance in the understanding of the phenomenon of transition from laminar to turbulent flow.

## 2. Salient features of the formulation

The salient features of the general formulation of the problem are outlined in this section. The expressions specific to Stuart’s (1960) theory, Watson’s (1960) theory and Reynolds & Potter’s (1967) theory are discussed in §3. This is followed in §§4 and 5 by a brief discussion on the solution for the  $\psi_{1n}$  functions and the numerical procedures adopted.

The two-dimensional Navier–Stokes equation in terms of the stream function  $\Psi$  is given as follows:

$$\frac{\partial}{\partial t}(\nabla^2\Psi) + \frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\Psi) - \frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\Psi) - \frac{1}{R}\nabla^4\Psi = 0, \quad (1)$$

where  $R$  is the Reynolds number.

The stream function  $\Psi$  is decomposed into a mean part  $\phi_0$  and a Fourier series of perturbations  $\phi_n$  as follows:

$$\Psi = \phi_0(y, t) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \phi_n \exp\{ni\alpha(x - c_r t)\}, \quad (2)$$

where  $\alpha$  is the spatial wavenumber in the downstream  $x$ -direction and  $c_r$  is the phase velocity according to the linear theory, corresponding to a given  $\alpha$  and  $R$ . Further, for  $n < 0$ ,  $\phi_{-n} = \tilde{\phi}_n$ , where the tilde ( $\tilde{\phantom{x}}$ ) denotes the complex conjugate. All velocities are normalized with respect to the channel centreline velocity, all distances are normalized with respect to the half-width of the channel, and the lateral distance  $y$  is measured from the centreline of the channel.

The mean velocity  $\bar{u}$  changes from its laminar value of  $\bar{u}_l = 1 - y^2$  owing to nonlinear effects. The expression for  $\bar{u}$  is given as

$$\bar{u} = \phi'_0 = 1 - y^2 + \sum_{n=1}^{\infty} |A|^{2n} F_n(y), \quad (3)$$

where primes denote differentiation with respect to  $y$ ,  $A$  is the amplitude associated with the eigen function of the linear problem, and  $F_n(y)$  are called the mean-motion distortion functions.

To enable separation between time and space variables, any disturbance harmonic  $\phi_p$  can be expressed as

$$\phi_p = \sum_{d=0}^{\infty} A^d |A|^{2d} \psi_{pd}, \quad (4)$$

where  $A = A(t)$  and  $\psi_{pd} = \psi_{pd}(y)$ .

The equation for amplitude growth is given by the Stuart–Landau series as

$$\frac{dA}{dt} = \alpha c_1 A + i\alpha A \sum_{n=1}^{\infty} K_n |A|^{2n}, \quad (5)$$

where the  $K_n$  are called the Landau coefficients. Since  $A$  is assumed to be complex, the real amplitude is given by the modulus of  $A$ , so that

$$\frac{d|A|^2}{dt} = 2\alpha c_i |A|^2 - 2\alpha |A|^2 \sum_{n=1}^{\infty} K_{ni} |A|^{2n}, \quad (6)$$

where the subscript  $i$  in  $K_{ni}$  indicates the imaginary part of  $K_n$ .

Substitution of (2)–(6) in (1) yields, after some algebra, the differential equation for any  $\psi_{pd}$  function and the differential equation for any mean-motion distortion function  $F_n$ , which are given below as (7) and (8) respectively:

$$\begin{aligned} L(p\alpha) \psi_{pd} = & \frac{2ic_1 d}{p} g_{pd} - \sum_{e=1}^d \left[ K_e + \frac{2i(d-e)}{p} K_{ei} \right] g_{p,d-e} - \sum_{m=1}^d [g_{p,d-m} F_m - \psi_{p,d-m} F_m''] \\ & - \frac{1}{p} \sum_{n=1}^{p-1} \sum_{e=0}^d [n\psi'_{p-n,d-e} g_{ne} - (p-n)\psi_{p-n,d-e} g'_{ne}] \\ & + \frac{1}{p} \sum_{n=1}^d \sum_{e=0}^{d-(e+n)} [n\psi'_{n+p,d-e-n} \tilde{g}_{ne} + (n+p)\psi_{n+p,d-e-n} \tilde{g}'_{ne} \\ & - (n+p)\tilde{\psi}'_{ne} g_{n+p,d-e-n} - n\tilde{\psi}_{ne} g'_{n+p,d-e-n}], \end{aligned} \quad (7)$$

$$\begin{aligned} 2n\alpha c_i F_n - \frac{1}{R} F_n'' = & 2\alpha \sum_{p=1}^{\infty} (n-p) K_{pi} F_{n-p} \\ & + i\alpha \frac{d}{dy} \sum_{d=0}^{\infty} \sum_{e=0}^{n-(d+e)} (n-d-e) [\psi_{n-d-e,d} \tilde{\psi}'_{n-d-e,e} - \psi'_{n-d-e,d} \tilde{\psi}_{n-d-e,e}]. \end{aligned} \quad (8)$$

In (7),  $L(p\alpha)$  is an Orr–Sommerfeld-type operator given as follows:

$$L(p\alpha) [ \ ] = (\bar{u}_r - c) \left( \frac{d^2}{dy^2} - p^2 \alpha^2 \right) [ \ ] - \bar{u}_r'' [ \ ] + \frac{i}{p\alpha R} \left( \frac{d^4}{dy^4} - 2p^2 \alpha^2 \frac{d^2}{dy^2} + p^4 \alpha^4 \right) [ \ ], \quad (9)$$

where  $c = c_r + ic_i$  is the complex phase velocity according to linear theory. Also  $g_{pd}$  is the vorticity component corresponding to  $\psi_{pd}$ , and is given as  $g_{pd} = \psi_{pd}'' - p^2 \alpha^2 \psi_{pd}$ .

The equations for the  $\psi_{pd}$  functions are therefore of the form

$$L(p\alpha) \psi_{pd} = N_{pd}, \quad (10)$$

with suitable boundary conditions as follows:

$$\psi_{pd} = \psi'_{pd} = 0 \quad (y = \pm 1), \quad (11a)$$

$$\psi'_{pd} = \psi'''_{pd} = 0 \quad (y = 0, \text{ } p \text{ odd}), \quad (11b)$$

$$\psi_{pd} = \psi''_{pd} = 0 \quad (y = 0, \text{ } p \text{ even}). \quad (11c)$$

For the  $F_n$  functions given in (8), the boundary conditions are

$$F_n = 0 \quad (y = \pm 1), \quad (12a)$$

$$F_n' = 0 \quad (y = 0). \quad (12b)$$

The boundary conditions given by (11) specify a disturbance mode which is known to become unstable, according to linear theory, in the region inside the neutral curve in the  $(\alpha, R)$ -plane.

Further, the eigenfunction  $\psi_1$  has to be suitably normalized at the channel centreline. Also, a typical method of obtaining the Landau coefficients  $K_n$  is to impose

the solvability condition on the equations for  $\psi_{1n}$ . A typical equation for  $\psi_{1n}$  is of the form

$$L(\alpha)\psi_{1n} = N_{1n} = T_{1n} - K_n g_1, \quad (13)$$

from which, following Stuart (1960), the  $K_n$  coefficients can be obtained as

$$K_n = \frac{\int_0^1 \theta T_{1n} dy}{\int_0^1 \theta g_1 dy}, \quad (14)$$

where  $\theta$  is the adjoint to the eigenfunction  $\psi_1$ . In the Watson method,  $T_{1n}$  in (13) contains the  $\psi_{1n}$  function itself. Therefore the solution is somewhat more complicated than in the RP method. Further, a suitable normalization has also to be prescribed for the  $\psi_{1n}$  function. This was done in the present work by excluding the  $\psi_1$  content in the  $\psi_{1n}$  functions. Some details regarding solution by Watson's method and the normalization of  $\psi_{1n}$  are discussed in §4.

### 3. Specific theories and their respective formulations

The general formulation outlined in §2, culminating in (7) and (8), may be called the Watson formulation, in which no simplifying assumptions have been made. For this reason, the Watson formulation is also called the 'true problem'. A serious difficulty regarding solution of the mean-motion equation (8) in Watson's method has been pointed out by Davey & Nguyen (1971). They mention that in the region  $c_1 < 0$  the mean-motion equation is likely to encounter singularities. This important point is discussed in §7.

In the method of Watson, the disturbance amplitude  $|A|$  is used as the small parameter, i.e.  $|A|^2$  is  $O(\epsilon)$ , and  $c_1$  is  $O(1)$ . With these assumptions, the Watson method should ordinarily be valid at points away from the neutral curve as well, provided that other difficulties like those in the case of the mean-motion equation mentioned earlier, are not encountered. With the assumption of  $|A|^2$  as  $O(\epsilon)$ , Watson's method is always asymptotically valid for  $|A| \rightarrow 0$ . Within the radius of convergence of the Stuart-Landau series, or in an extended region of convergence obtainable by using a method like the Shanks method, the Watson method can be used to predict  $dA/dt$  or  $d|A|^2/dt$ , subject to  $c_1 > 0$ . Also, if the equilibrium amplitude lies within the region of convergence or extended convergence, this can also be predicted by the Watson method, for  $c_1 > 0$ .

The Stuart (1960) formulation looks very similar to the Watson formulation, but there is an important difference. Stuart basically assumes  $c_1$  as the small parameter and not  $|A|$ , and the ordering of the magnitude of  $|A|$  is done in terms of  $c_1$ . Stuart has pointed out to us that equations corresponding to his approach are not correctly obtained by letting  $c_1$  approach zero in a general formulation of the type given by (7) and (8), except up to cubic-order terms in  $A$ . This is because, unless a suitable expansion is found for  $c_1$ , terms of like order of magnitude cannot be filtered out. An appropriate extension and subsequent development of Stuart's theory is the Stewartson & Stuart (1971) theory of development of a wavepacket from the nose of the neutral curve. Itoh (1977*a*) also suggests an improved version of Stuart's theory. We shall not dwell on Stuart's theory any further as it is outside the scope of the present work to investigate equations different from the form of (7) and (8). However, as a final comment on Stuart's work, it may be noted that the first Landau

coefficient  $K_1$  calculated by us using Stuart's (1960) cubic-order theory (which is correct) is numerically quite close to the  $K_1$  calculated by the equilibrium amplitude method of RP. This result is also implicitly corroborated in Pekeris & Shkoller's (1967) work, where they mention that 'the use of  $c_{1r} + ic_{1i}$  in (24) (equation for second harmonic), in place of  $c_{1i}$ , did not change  $|\beta|$  appreciably' (Pekeris & Shkoller 1967, p. 38).

The equilibrium-amplitude formulation of Reynolds & Potter (1967) attempts to model only the equilibrium state. With this restrictive end in view,  $d|A|^2/dt$  is put equal to zero at an early stage in the calculations irrespective of whether equilibrium actually exists or not. For this reason the method is also called the 'false problem'. *A priori* substitution of  $d|A|^2/dt = 0$  results in considerable simplification in the numerical work. In particular, the difficulties regarding the solution of the mean-motion equation for  $c_i < 0$  are removed. The formulation for this method is derivable from (7) and (8), in which the specific changes due to the equilibrium assumption are as follows.

The terms  $(2ic_i d/p)g_{pd}$  and  $-\Sigma_{e-1}^d (2i(d-e)/p)K_{ei}g_{p, d-e}$  drop out from the right-hand side of (7). Also, the term  $2n\alpha c_i F_n$  drops out from the left-hand side of (8), and the term  $2\alpha \Sigma_{p-1}^\infty (n-p)K_{pi}F_{n-p}$  drops out from the right-hand side of (8).

We next consider the Stuart-Landau series in the form given below, in order to formalize certain definitions:

$$S = \frac{1}{2\alpha|A|^2} \frac{d|A|^2}{dt} = c_i - \sum_{n=1}^{\infty} K_{ni}|A|^{2n}. \quad (15)$$

In Watson's theory, if the limit  $|A| \rightarrow 0$  is taken in (15) then the linear-theory relationship is retrieved. Further, within the radius of convergence of the series, (15) gives us the values of  $S$  corresponding to different levels of  $|A|$ , and, the 'zero-crossing' of  $S$  gives us the equilibrium state. The value of  $|A|$  for which  $S = 0$  is called the equilibrium amplitude. The RP theory is a special case of the Watson theory, in which the focus of attention is the zero-crossing of  $S$ . So long as one remains within the radius of convergence (or extended radius of convergence by using a method like the Shanks method) of (15), there is no restriction in applying Watson's theory or the RP theory to flows with or without a neutral curve, except for the following limitations. Watson's method cannot be used when  $c_i < 0$ , and the RP method is valid only for the equilibrium state and when the equilibrium state exists.

We conclude this section by mentioning that there is an important theory in the field due to Itoh (1974*a, b*, 1977*a, b*) in which an asymptotic theory using the amplitude as the small parameter is developed for a non-monochromatic system. Discussions on Itoh's theory are outside the scope of the present work, although we shall refer to Itoh's theory in a separate paper on pipe flow.

#### 4. The solution for the $\psi_{1n}$ functions

We next embark upon a discussion on the solution for the  $\psi_{1n}$  functions, which are a bit involved particularly for Watson's method. Our aim will be to arrive at a rational scheme of normalization of the  $\psi_{1n}$  functions, and thereafter to develop a suitable solution procedure that can be matched to the general numerical procedures adopted herein.

To consider the above questions the salient features of Watson's (1960) own elegant solution for  $\psi_{11}$  are discussed next. Attention is drawn to pp. 375-382 of Watson's (1960) paper, i.e. his equations (1.1.16)-(2.1.26). In the ensuing discussion, as far as

possible, the notation of Watson will be followed. Consider the equation for  $\psi_{11}$ , which is given as follows:

$$L(\alpha) \psi_{11} - 2ic_1(\psi_{11}'' - \alpha^2 \psi_{11}) = \frac{ia_1}{\alpha} (\psi_1'' - \alpha^2 \psi_1) + g_{11}, \tag{16}$$

where  $a_1$  is Watson's form of the first Landau coefficient and  $g_{11}$  represents the nonlinear forcing terms. Watson envisages the solution of  $\psi_{11}$  as the sum of the solutions of the following two systems A and B:

*system A* 
$$L(\alpha) \psi_{11} - 2ic_1(\psi_{11}'' - \alpha^2 \psi_{11}) = g_{11}; \tag{17}$$

*system B* 
$$L(\alpha) \psi_{11} - 2ic_1(\psi_{11}'' - \alpha^2 \psi_{11}) = \frac{ia_1}{\alpha} (\psi_1'' - \alpha^2 \psi_1). \tag{18}$$

Watson proposes the solution of system A as an asymptotic series in  $c_1$  as follows:

$$(\psi_{11})_A = \frac{1}{c_1} \psi_{11}^{(-1)} + \psi_{11}^{(0)} + c_1 \psi_{11}^{(1)} + \dots \tag{19}$$

Substitution of (19) in (17) leads to the following set of differential equations:

$$L(\alpha) \psi_{11}^{(-1)} = 0, \tag{20}$$

$$L(\alpha) \psi_{11}^{(0)} = g_{11} + 2i(\psi_{11}^{(-1)''} - \alpha^2 \psi_{11}^{(-1)}), \tag{21}$$

$$\left. \begin{aligned} L(\alpha) \psi_{11}^{(1)} &= 2i(\psi_{11}^{(0)''} - \alpha^2 \psi_{11}^{(0)}) \\ &\vdots \end{aligned} \right\} \tag{22}$$

The solution for  $\psi_{11}^{(-1)}$  from (20) is given as

$$\psi_{11}^{(-1)} = \lambda_0 \psi_1. \tag{23}$$

The arbitrary constant  $\lambda_0$  is determined from the solvability condition of (21):

$$\int_0^1 \theta [g_{11} + 2i\lambda_0(\psi_1'' - \alpha^2 \psi_1)] dy = 0, \tag{24}$$

where  $\theta$  is the adjoint of  $\psi_1$ . Next, the solution for  $\psi_{11}^{(0)}$  is given from (21) as

$$\psi_{11}^{(0)} = \lambda_1 \psi_1 + \phi_{11}^{(0)}, \tag{25}$$

where  $\lambda_1$  is an arbitrary constant and  $\phi_{11}^{(0)}$  is a certain particular integral of (21) having the value  $\phi_{11}^{(0)} = 0$  at  $y = 0$ , i.e. at the channel centreline. Again, the arbitrary constant  $\lambda_1$  in (25) is determined from the solvability condition of the equation for  $\psi_{11}^{(1)}$ , i.e. the first equation of (22). This gives

$$\int_0^1 \theta [(\phi_{11}^{(0)''} - \alpha^2 \phi_{11}^{(0)}) + \lambda_1(\psi_1'' - \alpha^2 \psi_1)] dy = 0. \tag{26}$$

Thereafter, the solution for  $\psi_{11}^{(1)}$  is given from the first equation of (22) as

$$\psi_{11}^{(1)} = \lambda_2 \psi_1 + \phi_{11}^{(1)}, \tag{27}$$

where again  $\phi_{11}^{(1)}$  is a certain particular integral with  $\phi_{11}^{(1)} = 0$  at  $y = 0$ , and  $\lambda_2$  is an arbitrary constant to be determined as before from the solvability condition for  $\psi_{11}^{(2)}$ .

The procedure is thus continued, and the final solution for  $(\psi_{11})_A$  is obtained as

$$(\psi_{11})_A = \left[ \frac{\lambda_0}{c_1} + \lambda_1 + c_1 \lambda_2 + \dots \right] \psi_1 + \phi_{11}^{(0)} + c_1 \phi_{11}^{(1)} + \dots, \tag{28}$$

where it is to be remembered that all the particular integrals  $\phi_{11}^{(n)}$  are even functions and have the value zero at the channel centreline.

Next the solution for system B (18) is considered. This is given as

$$(\psi_{11})_B = -\left[\frac{a_1}{2\alpha c_1}\right]\psi_1. \tag{29}$$

The full solution of  $\psi_{11}$  is given as  $\psi_{11} = (\psi_{11})_A + (\psi_{11})_B$ . Watson pointed out that the only restriction on  $a_1$  is the boundedness of the quantity  $(\lambda_0/c_1 - a_1/2\alpha c_1)$  in the limit  $c_1 \rightarrow 0$ . Obviously such a wide latitude for the possible values of  $a_1$  leaves a great deal of arbitrariness in the problem. Watson's own suggestion was to put  $a_1$  as follows:

$$a_1 = 2\alpha\lambda_0. \tag{30}$$

The full solution for  $\psi_{11}$  is then given as

$$\psi_{11} = [\lambda_1 + c_1\lambda_2 + \dots]\psi_1 + \phi_{11}^{(0)} + c_1\phi_{11}^{(1)} + \dots \tag{31}$$

In the limit  $c_1 \rightarrow 0$ , the above solution becomes

$$\psi_{11} = \lambda_1\psi_1 + \phi_{11}^{(0)}. \tag{32}$$

Thus it is seen that on making Watson's choice for  $a_1$ , as given by (30), the  $\psi_{11}$  function does contain a part of the eigenfunction  $\psi_1$  even in the limit  $c_1 \rightarrow 0$ . This is normally no cause for worry, especially as Watson has pointed out that the addition of an arbitrary multiple of  $\psi_1$  to the solution for  $\psi_{11}$  leads only to a rearrangement of the series at higher orders, each separate arrangement giving a different set of Landau coefficients  $a_n$  ( $n > 1$ ).

For reasons that will be apparent later, the desirable choice of  $a_1$  should be one that eliminates the  $\psi_1$  content in  $\psi_{11}$ . To achieve this,  $a_1$  is expressed in the present procedure as an expansion in  $c_1$  as follows:

$$a_1 = a_{10} + c_1a_{11} + c_1^2a_{12} + \dots \tag{33}$$

It is now seen from (28) that the desired answer for  $a_1$  is obtained by making the following choices for the  $a_{1n}$  in (33):

$$a_{10} = 2\alpha\lambda_0, \quad a_{11} = 2\alpha\lambda_1, \quad \dots, \quad a_{1n} = 2\alpha\lambda_n. \tag{34}$$

The choices above give the final forms of  $\psi_{11}$  and  $a_1$  respectively as follows:

$$\psi_{11} = \phi_{11}^{(0)} + c_1\phi_{11}^{(1)} + c_1^2\phi_{11}^{(2)} + \dots, \tag{35}$$

$$a_1 = 2\alpha(\lambda_0 + c_1\lambda_1 + c_1^2\lambda_2 + \dots). \tag{36}$$

In the limit  $c_1 \rightarrow 0$ ,  $\psi_{11}$  and  $a_1$  become

$$\psi_{11} = \phi_{11}^{(0)}, \quad a_1 = 2\alpha\lambda_0. \tag{37}$$

Thus with the choice of  $a_1$  as in (33) and (34), the  $\psi_1$  content in  $\psi_{11}$  is eliminated. Also, since all the  $\phi_{11}^{(n)}$  are symmetric and have the value zero at the channel centreline,  $\psi_{11}$  is also symmetric and has the value zero at the channel centreline.

Now we discuss the reasons why the above normalization is desirable. First, the physical interpretation of the  $\psi_{1n}$  functions is that these are distortions of the eigenfunction  $\psi_1$ . Therefore it seems natural to exclude the  $\psi_1$  content in  $\psi_{1n}$  to avoid redundancy in the series.

The second reason is much more important. There is a difficulty in interpreting

the 'growth rate' and 'equilibrium amplitude' if the  $\psi_1$  content is not excluded in the  $\psi_{1n}$ . Recalling that the fundamental disturbance  $\phi_1$  is given as

$$\phi_1 = A\psi_1 + A|A|^2\psi_{11} + \dots, \quad (38)$$

one can easily see that unless the  $\psi_1$  content in the  $\psi_{1n}$  is excluded, the 'amplitude' of  $\psi_1$  is *not*  $A$ , but something else. For instance, based on Watson's original choice, see (30) and (31), the 'effective amplitude'  $A_{\text{eff}}$  of  $\psi_1$  is given as

$$A_{\text{eff}} = A \left[ 1 + \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} c_1^{p-1} \lambda_p^{(n)} |A|^{2n} \right], \quad (39)$$

where the superscript ( $n$ ) corresponds to the contribution respectively from each of the  $\psi_{1n}$ . Thus the interpretation of 'growth rate' and 'equilibrium amplitude' has to be done in terms of  $A_{\text{eff}}$  and *not*  $A$ ; i.e.  $(1/A_{\text{eff}})dA_{\text{eff}}/dt$  gives the growth rate, and the modulus form of the same expression when equated to zero serves to calculate the equilibrium amplitude. Thus working with such a cumbersome expression as in (39) is best avoided, especially as  $A_{\text{eff}}$  and  $A$  become the same when the  $\psi_{1n}$  are normalized with the  $\psi_1$  content excluded in these.

For the same reasons as before, the  $\psi_{1n}$  functions in the RP method were also normalized in the same way in the present calculations. It needs also to be mentioned that Herbert (1980) suggested that the  $\psi_{1n}$  functions should be normalized to have the value zero at the channel centreline, both in the method of RP and of Watson. Herbert, however, assigned no particular reason for this choice except for some numerical convenience. Accidentally, his choice fulfils the important condition of the exclusion of the  $\psi_1$  content in the  $\psi_{1n}$ .

Having settled the important question of the normalization of the  $\psi_{1n}$  functions, we turn our attention towards developing a convenient solution procedure for the  $\psi_{1n}$  functions. We rewrite (16) as

$$L(\alpha)\psi_{11} = 2ic_1(\psi_{11}'' - \alpha^2\psi_{11}) + g_{11} + \frac{ia_1}{\alpha}(\psi_1'' - \alpha^2\psi_1). \quad (16a)$$

Let us apply the solvability condition for (16a) as

$$\int_0^1 \theta \left[ 2ic_1(\psi_{11}'' - \alpha^2\psi_{11}) + g_{11} + \frac{ia_1}{\alpha}(\psi_1'' - \alpha^2\psi_1) \right] dy = 0. \quad (40)$$

The technique of solution adopted hereafter is to iterate a few times between (16a) and (40). At the start,  $\psi_{11}$  is assumed to be identically zero in (40) and also in the right-hand side of (16a). The first approximation to  $a_1$  is then obtained from (40) (which is the same as  $2\alpha\lambda_0$ , see (24) and (34)), and this is put into the right-hand side of (16a). Next, (16a) is solved to obtain the first approximation for  $\psi_{11}$ , remembering that  $\psi_{11}$  has to be normalized with the value zero at the centreline. This first approximation for  $\psi_{11}$  is the same as  $\phi_{11}^{(0)}$  (see (21)). The process is thus continued to a desired level of convergence. It may be noted that every successive iteration of (16a) and (40) introduces corrections to  $\psi_{11}$  and  $a_1$  by amounts which are the *same* as the successive terms in the series given by (35) and (36) respectively. Thus the convergence rate of the procedure is identical with what would be obtained by successively solving the original differential equations of Watson, viz (20)–(22) and (24), (26) and so on. Nevertheless, the present procedure of iterating (16a) and (40) is much more convenient because the computer has to deal with the 'same' two equations, and there is therefore considerable saving in algebraic and programming labour. Moreover, except for the matter of iteration of (16a) and (40) the algebraic and numerical labour is virtually no different than in the case of the RP method,



although the theoretical ramifications in Watson's method are very much more. To sum up, we conveniently obtain the solutions for  $\psi_{1n}$  in Watson's formulation by *direct* use of (16*a*) and (40), and there is no need to solve separately the system of differential equations given by Watson for the solution of the  $\psi_{11}^{(n)}$  functions.

### 5. Certain computational aspects

The basic scheme of computation adopted is that of Thomas (1953), which is a finite-difference technique employing direct Gaussian elimination for the matrix solutions. The method of Thomas, although not necessarily amongst the most accurate for solving Orr–Sommerfeld-type equations, has the great advantage of numerical stability. It is also easy to control the order of errors in this method by choosing a suitable step size and by employing a Noumerov-type auxiliary function.

With reference to (7) and (8), it is seen that the total numbers of terms on the right-hand sides of the differential equations for  $\psi_{pd}$  and  $F_n$  become very large as the orders of  $p$ ,  $d$  or  $n$  become large. Nevertheless, initially for calculations up to  $K_3$ , the exact algebraic expressions on the right-hand sides, corresponding to each particular  $\psi_{pd}$  or  $F_n$ , were actually first worked out by hand and then fed into the computer. To calculate the higher-order  $K_n$ , a technique was developed whereby the algebra was worked out in the computer itself. This was achieved by storing the entire range of functions  $\psi_{pd}$  and  $F_n$ , along with their derivatives, in two huge third-order arrays of the type  $S(N, N, J)$ , and with the summations worked out in DO-loops. Once such a generalized procedure became operational in the computer, the chances of human error in the algebra were totally eliminated, especially after the initial calculations up to  $K_3$  were checked out by the generalized scheme.

In the present problem, the numerical procedure has three basic variants; viz the free solutions for  $\psi_1$  and  $\theta$ , the forced solutions for  $\psi_{pd}$  ( $p > 1$ ), and the solutions of the  $\psi_{1n}$  functions. The former two have been described in earlier papers (Thomas 1953; Pekeris & Shkoller 1967), and the solution for  $\psi_{1n}$ , along with a new method of obtaining  $K_n$ , will be described here. All the three situations referred to can be tackled by minor variations of the Gaussian elimination procedure.

We next discuss the solution procedure in brief, and with reference to Thomas's (1953) work and Pekeris & Shkoller's (1967) work. The original  $\psi_{pd}$  or  $F_n$  functions are converted, at the time of solution, to an auxiliary function  $g$  (see Thomas 1953) by means of a Noumerov-type transformation. Also, application of the finite-difference technique to an Orr–Sommerfeld-type operator like  $L(p\alpha)$ , results in this being converted into a pentadiagonal band matrix  $[A_{ij}]$ . Thus a typical differential equation for  $\psi_{pd}$  becomes, in equivalent matrix form,

$$[A_{ij}][g_j] = [P_i] \quad (i, j = 1, 2, 3, \dots, N+1), \tag{41}$$

where  $[A_{ij}]$  is the matrix equivalent of the operator  $L(p\alpha)$ ,  $[g_j]$  is the auxiliary function vector, and the vector  $[P_i]$  represents the right-hand-side forcing terms. The indices  $i$  and  $j$  refer to stations along the ordinate  $y$ ;  $i, j = 1$  representing  $y = 0$ , and  $i, j = N+1$  representing  $y = 1$ , where  $N$  is the number of steps being used in the finite-difference procedure. In the present calculations  $N$  was taken as 100, and the basic truncation errors were thus kept to  $O(10^{-8})$ . Calculations were performed on an ICL 2960 computer using double-precision arithmetic.

In the case of the free solutions like  $\psi_1$  and  $\theta$ , the right-hand column vector  $[P_i]$  is identically zero, since there is no forcing term. Thus a solution is only possible if  $\det [A_{ij}]$  is zero. This condition serves to calculate the eigenvalue (see Thomas 1953).

Upon Gaussian elimination of (41), the equation specifically at  $i = 1$  becomes

$$A'_{11}g_1 = P'_1, \quad (42)$$

where the prime denotes the altered value after Gaussian elimination. Thus, if  $P'_1$  is zero, as is in the case of free solutions, then it is necessary for the element  $A'_{11}$  to be zero. Further, if the element  $A'_{11}$  is zero then  $\det[A_{ij}]$  is also zero. Actually, the element  $A'_{11}$  does become zero when a correct eigenvalue for  $c$ , corresponding to a given  $\alpha$  and  $R$ , has been kept in the equations. Thereafter, an arbitrary value can be assigned to  $g_1$ , which means that a normalization has to be specified for  $\psi_1$  or  $\theta$ .

In the case of forced solutions for  $\psi_{pd}$  ( $p > 1$ ),  $P'_1$  in (42) is non-zero, since the vector  $[P_i]$  is non-zero. Also,  $A'_{11}$  is non-zero, and thus there is no difficulty in proceeding with the solution.

In the case of the  $\psi_{1n}$  functions there is a problem. The operator  $L(\alpha)$  for  $\psi_{1n}$  is identical with that for  $\psi_1$ . Thus  $[A_{ij}]$  and the element  $A'_{11}$  are also identical, the latter having the value zero. However, the forcing vector  $[P_i]$  is not zero in case of  $\psi_{1n}$ . Thus, with reference to (42), it is seen that a solution for the  $\psi_{1n}$  is possible only if the element  $P'_1$  (i.e. the first element in  $[P'_i]$ , where  $[P'_i]$  is the altered  $[P_i]$  after Gaussian elimination) is zero. However, the remaining elements in  $[P'_i]$  for  $i > 1$  need not be zero. Thus, in this matrix method, the solvability condition for the equation for  $\psi_{1n}$  is the requirement that  $P'_1$  should be zero. This condition actually gives us a new method for obtaining  $K_n$ , as explained next. The vector  $[P_i]$  for the  $\psi_{1n}$  function can actually be decomposed as  $[P_i] = [Q_i] + K_n[S_i]$ . Therefore, after Gaussian elimination,  $P'_1 = Q'_1 + K_n S'_1$ . Thus, if  $P'_1$  is to be zero, then  $K_n$  is determined as the ratio  $K_n = -Q'_1/S'_1$ . However, the usual way of determining  $K_n$  is by the adjoint method as in (14). It was found that the numerical values of  $K_n$ , by the matrix method and the adjoint method, were in agreement up to at least four significant figures. This served as a cross-check on the numerical work.

Finally, it is also seen from (42) that a normalization has to be specified for the  $\psi_{1n}$  functions as well. It is easy to see from (41) and (42) that the  $\psi_1$  content in  $\psi_{1n}$  is excluded if  $\psi_{1n}$  is normalized as  $\psi_{1n} = 0$  at  $y = 0$ .† This is in agreement with the theory discussed in §4.

## 6. Convergence of the Stuart–Landau series and determination of the equilibrium amplitude

### 6.1. The ‘correct sum’ of the Stuart–Landau series

The key equation in any method based on the Stuart–Watson formalism is the Stuart–Landau series given by (6) or (15). We begin our discussions specifically with reference to (15), in which the equilibrium state corresponds to the zero crossing of  $(1/2\alpha|A|^2)d|A|^2/dt$ , i.e. at  $S = 0$ . Thus, to know the equilibrium state, one must be able to obtain the ‘correct sum’ (to be defined subsequently) for  $S$  at different levels of  $|A|$ , and then one would be in a position to know the value of  $|A|$  corresponding to the zero-crossing of  $S$ , i.e. the equilibrium amplitude. The matter of ‘correct sum’ for  $S$  is very important, and is intimately connected with the convergence of the series. The discussions in this regard in this section are with reference to the RP method only, although certain of the general features of this discussion will be useful in §7 as well, where the Watson method and the RP method are compared.

† Imagine that the auxiliary function  $g$  is not being used. Then substitute  $\psi_1$  and  $\psi_{1n}$  respectively in place of  $g$  in both (41) and (42), and work out the matrix solutions for  $\psi_1$  and  $\psi_{1n}$ .

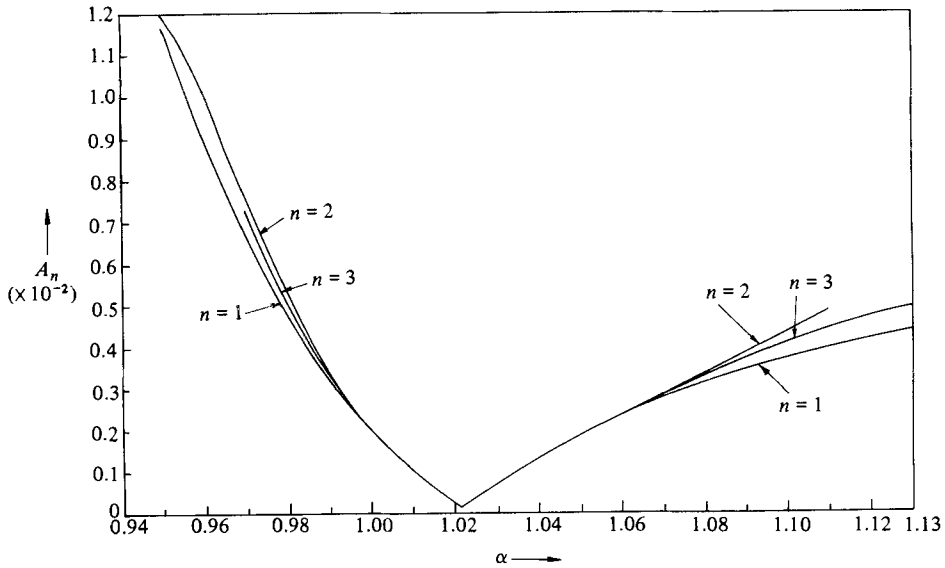


FIGURE 1. Variation of  $A_n$  with wavenumber  $\alpha$ ,  $R = 5774$ .  $A_1, A_2, A_3$  are equilibrium amplitudes calculated from the direct sum of the series (15) up to  $K_1, K_2, K_3$  respectively.

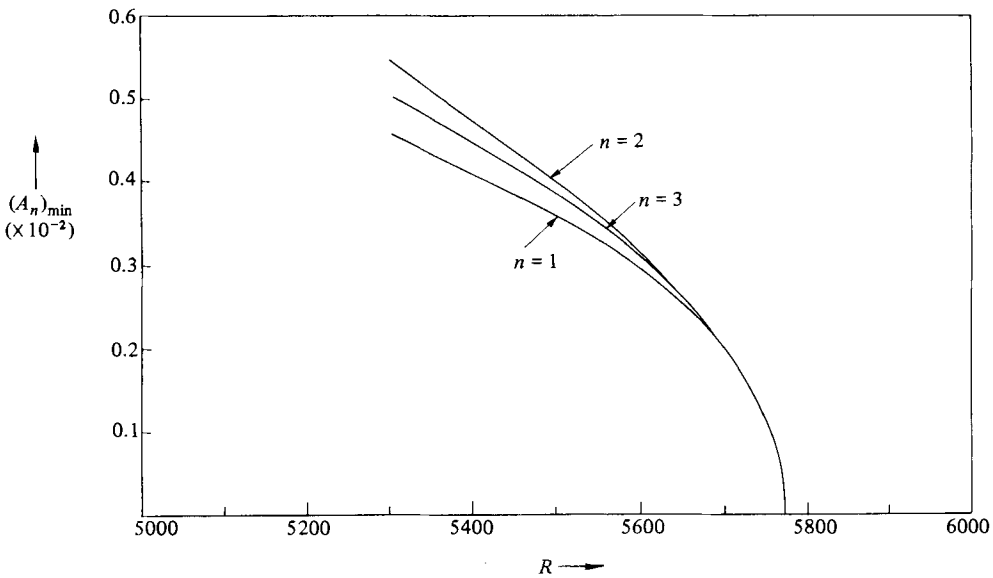


FIGURE 2. Variation of  $(A_n)_{\min}$  with Reynolds number  $R$ .  $(A_n)_{\min}$  is the minimum value of  $A_n$  for different  $\alpha$ , with a given  $R$  and  $n$ .  $A_1, A_2, A_3$  are the equilibrium amplitudes calculated from the direct sum of the series (15) up to  $K_1, K_2, K_3$  respectively.

First the region near the nose of the neutral curve is considered. Here, since  $c_i$  is small, the magnitude of the equilibrium amplitude (henceforth to be referred to as  $A_e$ ) is small. Thus for  $|A| \leq A_e$ , the series in (15) converges rapidly. Here, since the series converges, the 'correct sum' for  $S$  is given as the sum to infinity of the series, and, because the series converges rapidly, the correct sum is virtually the same as the numerical sum of the series up to a finite number of terms. Evidence of rapid convergence in this region is seen in figures 1 and 2 and also in Herbert's (1980) figure 2.

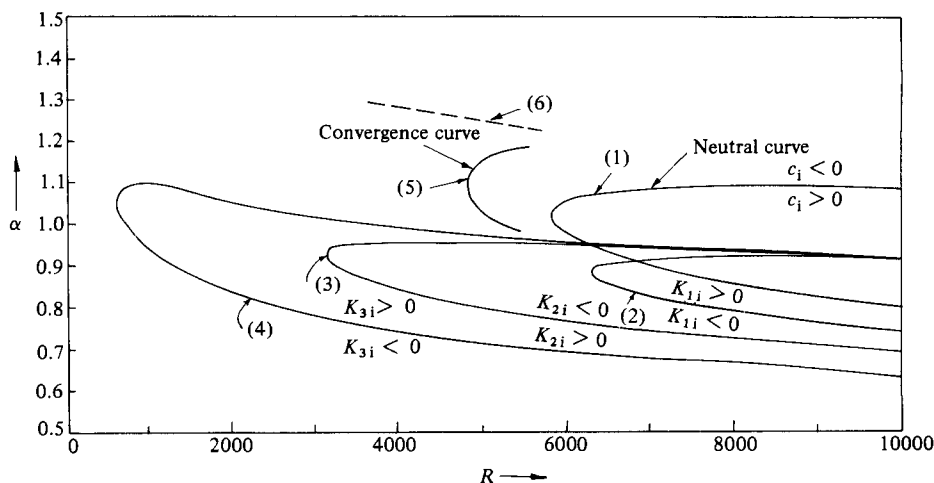


FIGURE 3. Stability diagram showing curves of (1)  $c_i = 0$ , (2)  $K_{1i} = 0$ , (3)  $K_{2i} = 0$ , (4)  $K_{3i} = 0$ , in the  $(\alpha, R)$ -plane. Also (5) shows the 'convergence curve', and (6) shows the line across which the higher-order  $K_{ni}$  ( $n > 7$ ) change sign.

For regions away from the neutral curve, the rate of convergence of the series in (15) becomes slower, and the series may even diverge. Interest is actually greater in the subcritical region, as this region is of practical importance. Before taking up the matter of the correct sum of the series, it seems that a preliminary investigation of the nature of the  $K_n$  coefficients, particularly the sign changes in  $K_{ni}$ , is desirable. This is shown in figure 3, which shows the points where important changes in signs take place in the various  $K_{ni}$ . In figure 3, the useful subcritical region was found to lie in between the upper limb of the  $K_{3i} = 0$  curve and the dashed line in the figure along which the higher-order  $K_{ni}$  change sign. Within this region defined, it was found that the  $K_{ni}$  are of alternating sign, and so are the real parts of  $K_n$ . Besides, outside this region also, the  $K_{ni}$  coefficients were found to alternate eventually in sign.

We now concentrate our attention to the useful subcritical region, as defined earlier, in which the  $K_{ni}$  are of consistently alternating sign. A typical point is at  $\alpha = 1.15$  and  $R = 5000$ , for which  $A_e$  is close to, but within, the radius of convergence  $r_c$  of the series. Nineteen  $K_n$  coefficients were calculated for this point, and the results are given in table 1. The  $\psi_1$  function was normalized as  $\psi_1 = (\alpha R)^{-\frac{1}{2}}$  at  $y = 0$ , in order to keep the magnitudes of  $K_n$  within bounds for the computer. The idea for this choice of normalization was obtained from the work of Davey & Nguyen (1971) on pipe flow. Their discussion on normalization and scales for the wall mode is found to be apt for the present case of plane flow as well. To obtain values of  $K_n$  corresponding to the normalization  $\psi_1 = 1$  at  $y = 0$ , the values for  $K_n$  in table 1 have to be multiplied by  $(\alpha R)^{\frac{1}{2}}$ . However, the equilibrium amplitudes given in table 1 and elsewhere correspond to the normalization  $\psi_1 = 1$  at  $y = 0$ .

We next proceed to evaluate the direct sum of the series  $S$  at  $\alpha = 1.15$  and  $R = 5000$ , up to different number of terms, and for different levels of amplitude. The result is shown in figure 4, and this is truly revealing. It appears that, even at levels of amplitude 15% less than the radius of convergence, the partial sums of the series up to odd orders of  $K_{ni}$  and even orders of  $K_{ni}$  are substantially different. Moreover, the partial sums up to even orders in  $K_{ni}$  show no equilibrium amplitude at all. The pattern of the diagram is the same at other points in the subcritical  $(\alpha, R)$ -plane as well, except that for points closer to the neutral curve the abscissa  $S = 0$  is higher

$n$	$K_n$		$A_n$ equilibrium amplitude based on direct sum for $S$ , (15), up to the $K_{n_i}$ term
	$K_{nr}$	$K_{ni}$	
1	$-0.2088 \times 10^{-2}$	$-0.1067 \times 10^{-2}$	0.005590
2	$0.1727 \times 10^{-3}$	$0.1440 \times 10^{-3}$	—
3	$-0.1736 \times 10^{-4}$	$-0.2079 \times 10^{-4}$	0.006348
4	$0.2044 \times 10^{-5}$	$0.3274 \times 10^{-5}$	—
5	$-0.2675 \times 10^{-6}$	$-0.5452 \times 10^{-6}$	0.006603
6	$0.3738 \times 10^{-7}$	$0.9424 \times 10^{-7}$	—
7	$-0.5438 \times 10^{-8}$	$-0.1672 \times 10^{-7}$	0.006725
8	$0.8088 \times 10^{-9}$	$0.3021 \times 10^{-8}$	—
9	$-0.1212 \times 10^{-9}$	$-0.5522 \times 10^{-9}$	0.006797
10	$0.1806 \times 10^{-10}$	$0.1017 \times 10^{-9}$	—
11	$-0.2639 \times 10^{-11}$	$-0.1878 \times 10^{-10}$	0.006846
12	$0.3715 \times 10^{-12}$	$0.3475 \times 10^{-11}$	—
13	$-0.4894 \times 10^{-13}$	$-0.6426 \times 10^{-12}$	0.006885
14	$0.5660 \times 10^{-14}$	$0.1187 \times 10^{-12}$	—
15	$-0.4633 \times 10^{-15}$	$-0.2188 \times 10^{-13}$	0.006917
16	$-0.1441 \times 10^{-16}$	$0.4021 \times 10^{-14}$	—
17	$0.1968 \times 10^{-16}$	$-0.7369 \times 10^{-15}$	0.006945
18	$-0.6488 \times 10^{-17}$	$0.1346 \times 10^{-15}$	—
19	$0.1666 \times 10^{-17}$	$-0.2450 \times 10^{-16}$	0.006970

TABLE 1. RP method;  $\alpha = 1.15$ ,  $R = 5000$ ,  $c = 0.2847 - 0.003433i$ . Normalization of  $\psi_1$ : for  $K_n$  coefficients  $\psi_1(0) = (\alpha R)^{-\frac{1}{2}}$ ; for amplitudes,  $\psi_1(0) = 1$ . Equilibrium amplitude by Shanks method  $A_e = 0.007387$ . Radius of convergence  $r_c = 0.00756$ . Note: to get the values of  $K_n$  corresponding to a normalization of  $\psi_1(0) = 1$ , multiply by  $(\alpha R)^{\frac{1}{2}n}$ .

up, and for points away the abscissa  $S = 0$  is lower down than in figure 4. From figure 4 it appears that, unless a means is found to accelerate the convergence of the series, even nineteen  $K_{ni}$  are not adequate to obtain a reasonable approximation of the sum to infinity for  $|A|$  of the order of  $A_e$ .

The expression 'correct sum' for  $S$ , hitherto used, needs to be defined at this stage. Within  $|A| < r_c$  this is obviously the sum to infinity of the series. For  $|A| < r_c$  the difficulty does not lie in being able to define the correct sum, but in obtaining its value, especially when  $|A|$  is of order  $r_c$ . It appears that, by use of the Shanks (1955) method, † the 'correct sum' for  $S$  can be defined and obtained in the divergent region of the series as well. To highlight those features of the Shanks method that are relevant to the present problem, we begin by considering an example. We consider the expression

$$f(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots \quad (43)$$

In the above expression the binomial expansion for  $f(z)$  gives rise to a series which has  $r_c = 1$ , and the expression has a pole at  $z = 1$ . For values of  $|z|$  close to and less than  $r_c$ , and for values of  $|z| > r_c$ , the binomial expansion shows an appreciable 'numerical transient' (as defined by Shanks). The transient is damped for  $|z| < r_c$ , but amplifies for  $|z| > r_c$ . However, the nonlinear transform  $e_1^m$  of Shanks filters out this numerical transient in either of the two cases, and gives the correct value for  $f(z)$ , irrespective of whether the binomial series is converging or diverging. Therefore

† The Shanks method was suggested to us by a referee of an earlier version of the paper.

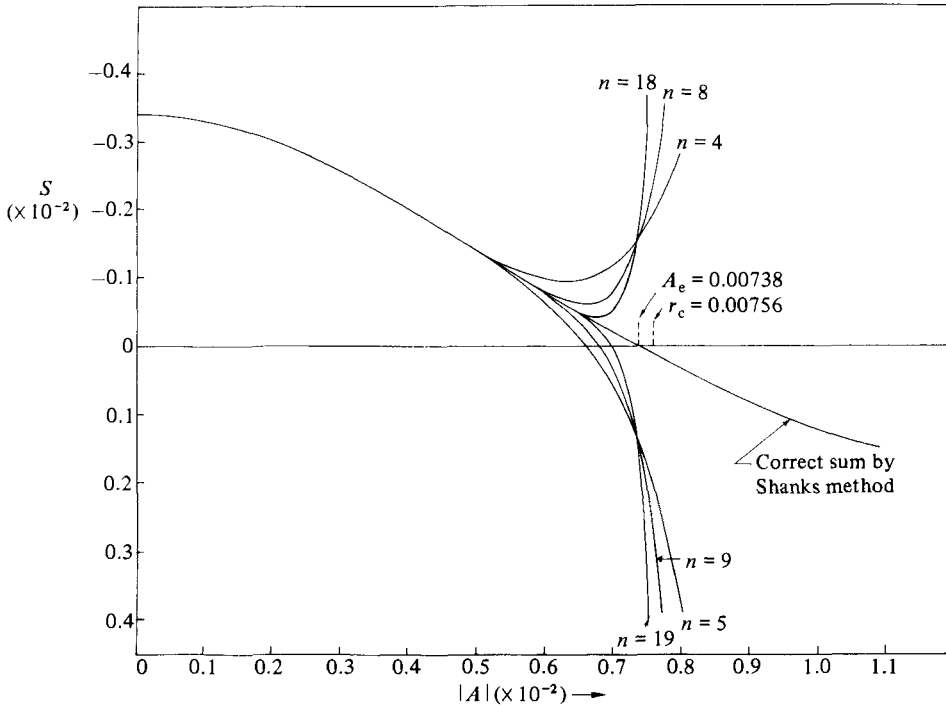


FIGURE 4. The sum of the series  $S$  (15) versus  $|A|$ . The direct sums of the series, up to different orders of  $n$  in  $K_n$ , are shown. Also shown is the correct sum of the series by the Shanks method. The equilibrium amplitude  $A_e$  and the radius of convergence  $r_c$  are also indicated. The results are for the point  $\alpha = 1.15$ ,  $R = 5000$ .

the 'correct sum' of a series is defined, following Shanks, as (i) the sum to infinity within the radius of convergence, and (ii) the analytical continuation of the sum of a convergent series in the divergent region of the series (see Shanks 1955, p. 12). The correct sum, for the divergent region of a series, is also called the antilimit of the series. Thus the Shanks method can be used as an accelerator of convergence in the poorly convergent region of a series, and as an inducer of convergence in the divergent region of a series.

In actuality, a series like the Stuart–Landau series, may have a more complicated form than the one corresponding to (43). For instance the series may contain more than one pole. For such a case, the Shanks method, using the  $e_1^m$  transform, actually gives the correct sum for values of  $|A|$  within the nearest singularity, and usually beyond the nearest singularity as well, provided certain conditions are met with (for details see Shanks 1955, Theorems III and IV). A notable exception, for the region beyond the nearest singularity, is when two singularities lie close to each other, in which case, the  $e_1^m$  transform fails but the  $e_2^m$  transform is appropriate. The present results were actually cross-checked by the  $e_2^m$  transform as well. However, the  $e_2^m$  transform itself may fail for a series with a single pole only, for instance when  $z = -1$  in the series corresponding to (43). This point is not mentioned by Shanks. Nevertheless, the  $e_1^m$  transform was found to be perfectly suitable for summing the Stuart–Landau series in both the convergent and the divergent regions.

The behaviour of any series purporting to describe a physical phenomenon requires careful attention at a pole, especially at the first one. The  $e_1^m$  transform 'correctly' sums a series at a pole, i.e. it gives the values, approaching  $+\infty$  or  $-\infty$ , on either

side of the pole. Yet the question remains as to whether this 'correct sum' is physically relevant or not. The answer, in most cases, would appear that this is not so, and the blowing up of a series at a pole is solely a limitation of the series itself, in failing to faithfully describe the physical phenomenon it purports to describe. The Shanks method provides no remedy for this inherent disadvantage of a series.

Fortunately, in the RP method, the Stuart–Landau series is not restricted by this above limitation at the nearest singularity. The series, being of alternating sign, has its nearest singularity on the negative  $|A|^2$  axis, which point is unimportant and physically unrealizable. Therefore, even in the divergent region of the series, the  $e_1^m$  transform sums the series efficiently and without controversy, until a singularity is encountered on the positive  $|A|^2$  axis. An instance of this is shown in figure 8. Thus, with reference to figure 4, with which the discussions began, the curve corresponding to the correct sum of the series is actually obtained by the  $e_1^m$  transform of Shanks. Also, the zero-crossing of this curve gives the correct value of the equilibrium amplitude. Indeed the curve in figure 4 corresponding to the correct sum is not much different for six terms or twenty terms in the series, for  $|A| \leq A_e$ ! This last result amply verifies the usefulness of the Shanks method in the present problem.

Before concluding this subsection, it is relevant to mention that the result for the sum of the series  $S$  is supposed to be physically meaningful only at the equilibrium amplitude in the 'false problem' of RP. However, as mentioned earlier, unless a reliable method is known by which the sum of the series can be correctly evaluated, for different levels of  $|A|$ , the equilibrium amplitude cannot also be found. It is in this context that the Shanks method proves to very useful. Further, it will be seen in §7 that in the supercritical region the Watson method and the RP method give substantially the same sums for  $S$ , for  $|A| \leq A_e$ .

## 6.2. Determination of the radius of convergence of the series

There is always interest in knowing the radius of convergence of the Stuart–Landau series, even though the nearest singularity is in the negative real axis of  $|A|^2$ , because, within the radius of convergence, the more conventional methods of series analysis are applicable. The standard way of determining the radius of convergence is by means of Domb–Sykes plots (cf. Van Dyke 1975), which is a refined form of the ratio test. Typical Domb–Sykes plots are given in figures 5 and 6. It is seen in these figures that for some points in the  $(\alpha, R)$ -plane a trend develops in these plots within terms up to  $K_9$ . For others, like those at  $\alpha = 1.10$  and  $\alpha = 1.15$  at  $R = 5000$ , terms up to about  $K_{19}$  are necessary. Herbert (1980) also mentions this difficulty. The method proves therefore to be very laborious in the present problem, as so many of the higher-order  $K_n$  have to be calculated. The results obtained from the Domb–Sykes plots are thereafter utilized to better effect to obtain a 'convergence curve' in the  $(\alpha, R)$ -plane, which is shown in figure 3. Within this curve, and the neutral curve, the Stuart–Landau series converges.

The method of obtaining the 'convergence curve' is shown in figure 7. Here, for different values of  $R$  corresponding to a given  $\alpha$ , the radii of convergence and the equilibrium amplitudes by Shanks method are plotted. The intersection of any such pair of plots gives a pair of values of  $\alpha$  and  $R$  corresponding to a point on the convergence curve.

The method of obtaining radius of convergence by Domb–Sykes plots is laborious, but convincing. Actually, there is a simpler way of doing so, by using the Shanks method. Recalling that the  $e_1^m$  transform 'correctly' sums the series at a pole, one has only to investigate the negative  $|A|^2$  region by the Shanks method and locate

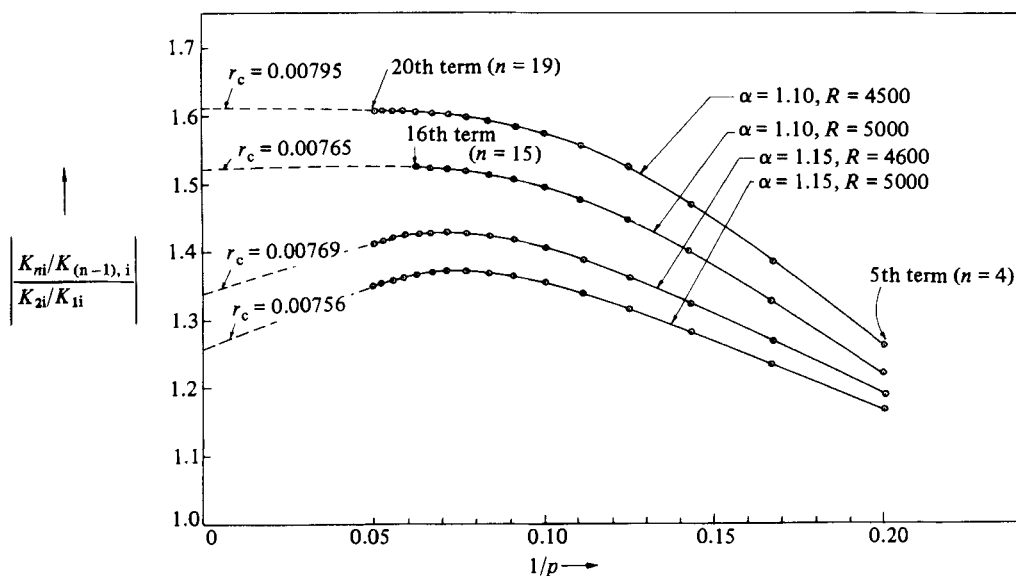


FIGURE 5. Domb-Sykes plots normalized by the ratio  $|K_{21}/K_{11}|$ . The order of terms in the series in (15) is given as  $p = n + 1$ . The radius of convergence is  $r_c$ . Typical points in the  $(\alpha, R)$ -plane, for which many of the higher-order  $K_{n1}$  are required before a trend develops in the Domb-Sykes plots, are shown in this figure.

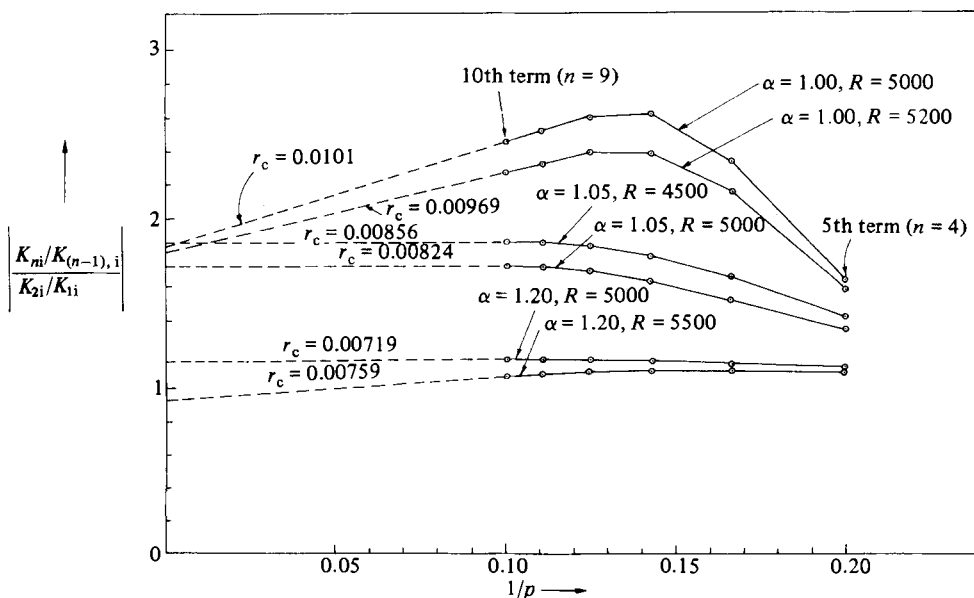


FIGURE 6. Domb-Sykes plots normalized by the ratio  $|K_{21}/K_{11}|$ . The order of terms in the series in (15) is given as  $p = n + 1$ . The radius of convergence is  $r_c$ . Typical points in the  $(\alpha, R)$ -plane, for which a trend develops within  $K_{n1}$  in the Domb-Sykes plots, are shown in this figure.



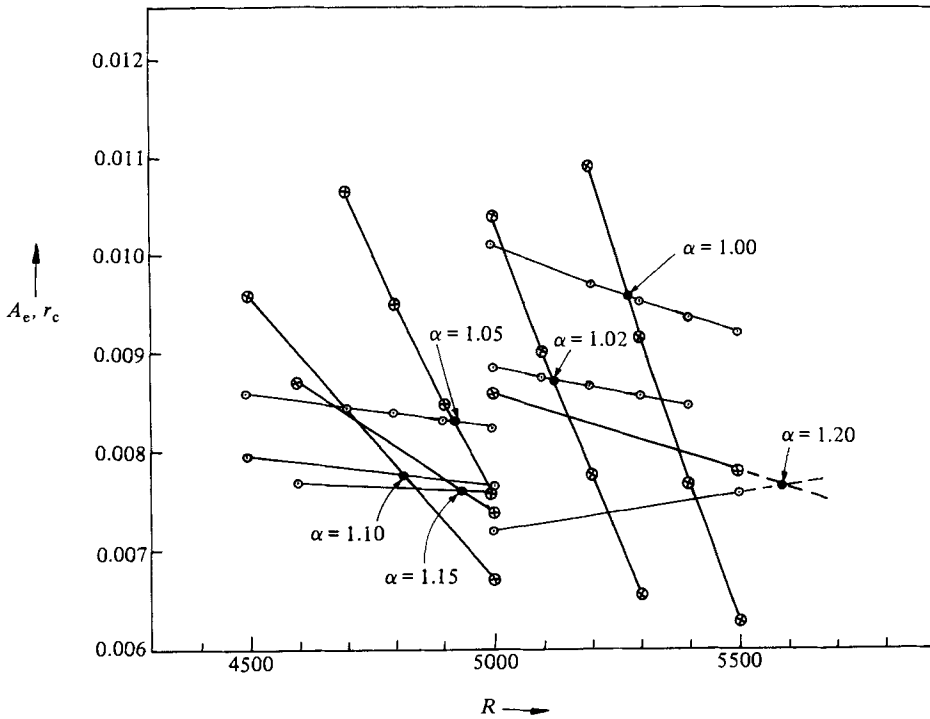


FIGURE 7. Method of obtaining the 'convergence curve' given in figure 3. Equilibrium amplitude  $A_e$  is obtained by the Shanks method. Radius of convergence  $r_c$  is obtained from Domb-Sykes plots. Variation of  $A_e$  and  $r_c$  with  $R$ , and for a given  $\alpha$ , are shown respectively by the lines  $\otimes$ — $\otimes$  and  $\odot$ — $\odot$ . Intersection of a pair of such lines, shown by  $(\bullet)$ , gives a point on the 'convergence curve'.

the nearest singularity. This can be achieved by using far fewer coefficients in  $K_n$ ; about seven seem to be adequate for most purposes.

### 6.3. An alternative convergence test and an alternative way of obtaining the correct sum of the series

It has been stated earlier that the correct sum of the series is obtained by the Shanks method. Nevertheless, it is in order to check out the calculation for the correct sum by a suitable alternative method. For this purpose we look at the results given in table 2, for the point  $\alpha = 1.04$  and  $R = 5000$ . In this table, one column shows the size of terms in the Stuart-Landau series, given by the sequence  $T_n$  defined by

$$T_0 = c_1, \quad T_n = -K_{ni}|A|^{2n} \quad (n = 1, 2, 3, \dots). \quad (44)$$

So long as  $|A| < r_c$ , the size of  $T_n$  will eventually diminish progressively with increasing  $n$ , although at a very slow rate for  $|A|$  close to  $r_c$ . Therefore, since  $T_n$  is a sequence with alternating signs,  $|T_{n-1}| > |T_n|$  and  $|T_n| \rightarrow 0$  for  $n \rightarrow \infty$  are sufficient conditions for the convergence of the  $T_n$  sequence. Existence of convergence can be further illustrated by the construction of the following sequence  $V_p$ :†

$$V_0 = \frac{1}{2}T_0, \quad V_p = \frac{1}{2}(T_{p-1} + T_p) \quad (p = 1, 2, 3, \dots). \quad (45)$$

It can be seen that the respective sums to infinity of the  $T_n$  and  $V_p$  sequences are

† The sequence  $V_p$  was suggested by a referee of an earlier version of the paper.

$n$	$K_{ni}$	$A_n$	$T_n$	$V_p$	$V_p^{(2)}$
			$T_0 = c_1$ $T_n = -K_{ni} A ^{2n}$	$V_0 = \frac{1}{2}T_0$ $V_n = \frac{1}{2}(T_{n-1} + T_n)$	$V_0^{(2)} = \frac{1}{2}V_0$ $V_n^{(2)} = \frac{1}{2}(V_{n-1} + V_n)$
0	—	—	$-0.1435 \times 10^{-2}$	$-0.7175 \times 10^{-3}$	$-0.3587 \times 10^{-3}$
1	$-0.3647 \times 10^{-3}$	0.006609	$0.2214 \times 10^{-2}$	$0.3896 \times 10^{-3}$	$-0.1639 \times 10^{-3}$
2	$0.3260 \times 10^{-4}$	—	$-0.1202 \times 10^{-2}$	$0.5063 \times 10^{-3}$	$0.4480 \times 10^{-3}$
3	$-0.3179 \times 10^{-5}$	0.007527	$0.7114 \times 10^{-3}$	$-0.2451 \times 10^{-3}$	$0.1306 \times 10^{-3}$
4	$0.3875 \times 10^{-6}$	—	$-0.5264 \times 10^{-3}$	$0.9248 \times 10^{-4}$	$-0.7632 \times 10^{-4}$
5	$-0.5485 \times 10^{-7}$	0.007725	$0.4524 \times 10^{-3}$	$-0.3699 \times 10^{-4}$	$0.2774 \times 10^{-4}$
6	$0.8356 \times 10^{-8}$	—	$-0.4185 \times 10^{-3}$	$0.1699 \times 10^{-4}$	$-0.1000 \times 10^{-4}$
7	$-0.1312 \times 10^{-8}$	0.007805	$0.3989 \times 10^{-3}$	$-0.9768 \times 10^{-5}$	$0.3611 \times 10^{-5}$
8	$0.2081 \times 10^{-9}$	—	$-0.3842 \times 10^{-3}$	$0.7363 \times 10^{-5}$	$-0.1203 \times 10^{-5}$
9	$-0.3309 \times 10^{-10}$	0.007858	$0.3709 \times 10^{-3}$	$-0.6649 \times 10^{-5}$	$0.3572 \times 10^{-6}$
			$\Sigma T_n =$	$\Sigma V_p =$	$\Sigma V_p^{(2)} =$
			$0.1822 \times 10^{-3}$	$-0.3240 \times 10^{-5}$	$0.8425 \times 10^{-7}$

TABLE 2. RP method;  $\alpha = 1.04$ ,  $R = 5000$ ,  $c = 0.2730 - 0.001435i$ . Normalization of  $\psi_1$ : for  $K_n$  coefficients,  $\psi_1(0) = (\alpha R)^{-\frac{1}{2}}$ ; for amplitudes,  $\psi_1(0) = 1$ . Equilibrium amplitude by Shanks method  $A_e = 0.008209$ ; radius of convergence  $r_c = 0.00827$ . Notes: (i) to get the values of  $K_n$  corresponding to a normalization of  $\psi_1(0) = 1$ , multiply by  $(\alpha R)^{\frac{1}{2}}$ ; (ii) value of  $|A|$  used in  $T_n$ ,  $V_p$  and  $V_p^{(2)}$  sequences is  $|A| = A_e$ ; (iii)  $A_n$  is the equilibrium amplitude based on direct sum of  $S$ , (15), up to the  $K_{ni}$  term.

the same, a fact which can be verified by using the Shanks method for either sequence. However, the sums up to a finite number of terms are not the same.

The rationale for introducing the  $V_p$  sequence is that if  $T_n$  is eventually alternating then  $V_p$  will also be eventually alternating. However, the numerical transient in the  $V_p$  sequence is found to be much less than in the  $T_n$  sequence, as is seen in table 2. Thus convergence is accelerated in the  $V_p$  sequence, and this helps to explore the existence of convergence with finer resolution, in the region of poor convergence of the  $T_n$  sequence.

The  $V_p$  transform of  $T_n$  can be taken a step further, to actually evaluate the sum of the series. This is achieved by repeating the  $V_p$  transform of the  $V_p$  series itself, and successively repeating the process. Each repetition of the  $V_p$  transform further suppresses the numerical transient, until, after  $m$  repetitions, the latest sequence  $V_p^{(m)}$  has a negligible numerical transient. Moreover, since the  $V_p^{(m)}$  sequence also eventually alternates, the size of the higher-order terms in this sequence is very small. Thus the sum of the  $V_p^{(m)}$  sequence is very close to the correct sum of the original series. The process is illustrated in table 2, where the  $V_p^{(2)}$  sequence is also shown. The original  $T_n$  sequence was constructed using  $|A| = A_e$ , where  $A_e$  was calculated by the Shanks method, which means that the sum to infinity of the  $T_n$  sequence should be zero. This is verified by the  $V_p^{(2)}$  sequence itself, the sum of the given ten terms of which is seen to be  $0.84 \times 10^{-7}$ .

The above method of obtaining the correct sum of the Stuart–Landau series is valid only for  $|A| < r_c$ , and on this count the Shanks method, which is applicable for  $|A| > r_c$  as well, overshadows this method. Incidentally the  $V_p^{(m)}$  method was also applied to the Leibnitz series for  $\pi$ , discussed by Shanks (1955) and Van Dyke (1975, pp. 202, 203). The sum up to ten terms of the  $V_p^{(2)}$  sequence itself gave the value of  $\pi$  correct to four significant figures.

6.4. A common mistake in calculating the equilibrium amplitude

A common mistake made in calculating the equilibrium amplitude, is to calculate a few terms in a Stuart–Landau-type series, to put  $S = 0$ , and thereafter to evaluate the value of  $|A|$  as the root of a polynomial equation.† The answer is wrong because the direct numerical sum up to a few terms in the series is not the same as the correct sum of the series unless the series converges rapidly. Moreover, this wrongly calculated value of  $A_e$  should not be substituted back into the series to investigate the convergence of the series. The extent of error that can occur is seen with reference to figure 4. In figure 4 let us consider a situation corresponding to a point far from the neutral curve, for which case the abscissa  $S = 0$  will be much lower down than in figure 4. For such a case, the incorrect ‘equilibrium amplitudes’, calculated as mentioned earlier, up to *odd* orders of  $K_{ni}$ , will seem to have values close to each other. This gives a fictitious notion of existence of convergence, even though terms up to *even* orders in  $K_{ni}$  do not give any root for  $A_e$ . The set of incorrect ‘equilibrium amplitudes’ are seen to be considerably lower than the correct equilibrium amplitude obtained by the Shanks method. It is further seen that, as one moves to the region of increasing divergence of the series, the incorrect set of ‘equilibrium amplitudes’ tend to acquire the value of the *radius of convergence of the series*, rather than the correct value of equilibrium amplitude. This is illustrated in table 3 for the results at  $\alpha = 1.10$  and  $R = 4500$ .

$n$	$K_n$		$A_n$
	$K_{nr}$	$K_{ni}$	equilibrium amplitude based on direct sum for $S$ , (15), up to the $K_{ni}$ term
1	$-0.1915 \times 10^{-2}$	$-0.7007 \times 10^{-3}$	0.006967
2	$0.1500 \times 10^{-3}$	$0.8166 \times 10^{-4}$	—
3	$-0.1493 \times 10^{-4}$	$-0.1040 \times 10^{-4}$	0.007820
4	$0.1760 \times 10^{-5}$	$0.1528 \times 10^{-5}$	—
5	$-0.2335 \times 10^{-6}$	$-0.2471 \times 10^{-6}$	0.007995
6	$0.3357 \times 10^{-7}$	$0.4236 \times 10^{-7}$	—
7	$-0.5112 \times 10^{-8}$	$-0.7517 \times 10^{-8}$	0.008030
8	$0.8117 \times 10^{-9}$	$0.1362 \times 10^{-8}$	—
9	$-0.1328 \times 10^{-9}$	$-0.2497 \times 10^{-9}$	0.008031
10	$0.2222 \times 10^{-10}$	$0.4611 \times 10^{-10}$	—
11	$-0.3774 \times 10^{-11}$	$-0.8554 \times 10^{-11}$	0.008026
12	$0.6480 \times 10^{-12}$	$0.1592 \times 10^{-11}$	—
13	$-0.1121 \times 10^{-12}$	$-0.2969 \times 10^{-12}$	0.008019
14	$0.1947 \times 10^{-13}$	$0.5547 \times 10^{-13}$	—
15	$-0.3392 \times 10^{-14}$	$-0.1038 \times 10^{-13}$	0.008013
16	$0.5917 \times 10^{-15}$	$0.1943 \times 10^{-14}$	—
17	$-0.1032 \times 10^{-15}$	$-0.3641 \times 10^{-15}$	0.008007
18	$0.1797 \times 10^{-16}$	$0.6824 \times 10^{-16}$	—
19	$-0.3122 \times 10^{-17}$	$-0.1279 \times 10^{-16}$	0.008002

TABLE 3. RP method;  $\alpha = 1.10$ ,  $R = 4500$ ,  $c = 0.2849 - 0.002869i$ . Normalization of  $\psi_1$ : for  $K_n$  coefficients,  $\psi_1(0) = (\alpha R)^{-\frac{1}{3}}$ ; for amplitudes,  $\psi_1(0) = 1$ . Equilibrium amplitude by Shanks method  $A_e = 0.009602$ ; radius of convergence  $r_c = 0.00795$ . Note: to get the value of  $K_n$  corresponding to a normalization of  $\psi_1(0) = 1$ , multiply by  $(\alpha R)^{\frac{1}{3}n}$ .

† An earlier unpublished version of this paper contained this mistake, resulting in the ‘convergence curve’ extending to much lower values of  $R$ .

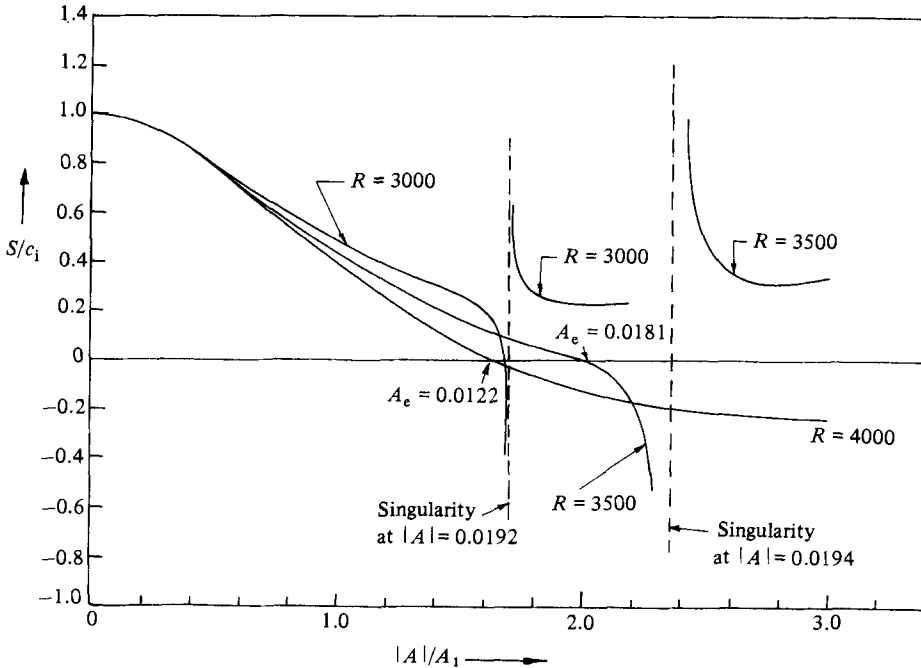


FIGURE 8.  $S/c_1$  versus  $|A|/A_1$  curves for  $\alpha = 1.15$  and  $R = 4000, 3500$  and  $3000$ .  $A_1$  is the rough value of the equilibrium amplitude based on  $K_{11}$ . The series in (15) was summed by the Shanks method. A singularity is seen to appear in the curves for  $R = 3500$  and  $3000$ .

Herbert's (1980) work seems not to have escaped the above error, as the analysis given by him for his results at  $\alpha = 1.12$  and  $R = 5000$ , seems to indicate. Zhou's (1982) work also appears to have a similar error. Reference to Zhou's results will be made in §6.6.

#### 6.5. The lower limit of $R$ up to which the equilibrium amplitude can be calculated

It has been seen that the Shanks method can be used to calculate the equilibrium amplitude in the divergent region of the series; and therefore for values of  $R$  less than that at the nose of the convergence curve given in figure 3. The question is, up to how low values of  $R$  this may be done. The point is considered in figure 8, where  $S/c_1$  versus  $|A|/A_1$  curves, obtained by the Shanks method, are given for  $\alpha = 1.15$  with  $R = 4000, 3500$  and  $R = 3000$  respectively. The curve for  $R = 4000$  shows a proper zero-crossing of  $S$ . For  $R = 3500$  it is seen that the series has a pole for a value of  $|A|$  slightly but distinctly higher than  $A_e$ . However, for  $R = 3000$ , this pole shifts to lower values of  $|A|/A_1$ , for which case the calculation of  $A_e$  becomes quite unreliable. It is therefore concluded that, by means of the procedures adopted here, it is not meaningful to calculate equilibrium amplitudes for values of  $R$  lower than  $R = 3500$ . Zhou (1982) has also arrived numerically at a similar conclusion.

#### 6.6. Calculation of the velocity fluctuations

The root-mean-square (r.m.s.) value of the  $u'$  fluctuation has been calculated by Herbert (1977) by the Fourier-truncation (FT) method, using  $N = 4$ . Also, Nishioka, Iida & Ichikawa (1975) have obtained curves for the maximum value  $u'_m$ , of the r.m.s.  $u'$  velocity fluctuation from their experimental work. This  $u'_m$  has been plotted versus the circular frequency  $\beta$ . In the present work also,  $u'$  has been calculated, and a plot

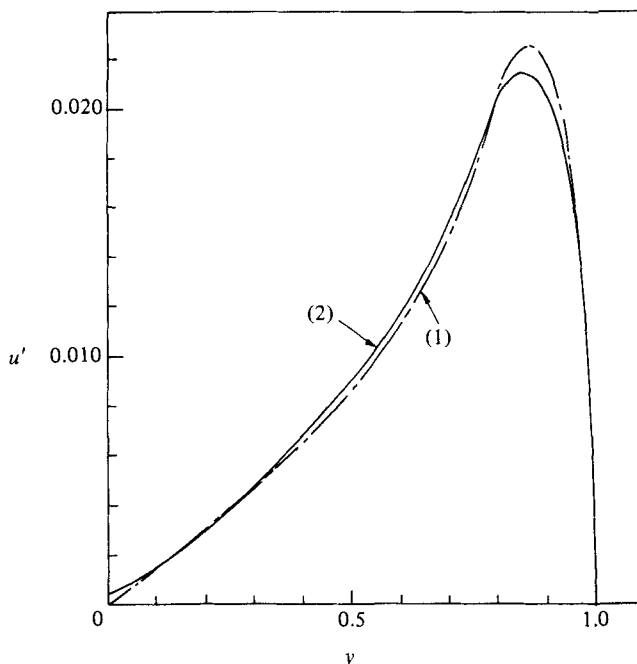


FIGURE 9. Distribution of the  $u'$  r.m.s. velocity for  $\alpha = 1.12$ ,  $R = 5000$ . Curve (1) shows the undistorted fundamental, and curve (2) shows the total  $u'$  r.m.s. velocity.

of a  $u'$  distribution is given in figure 9. For this purpose, the various series, like those in (4), were also summed by the Shanks method, for the real and imaginary parts separately. For the present work also, a  $u'_m$  versus  $\beta$  curve has been given in figure 10 ( $\beta = \alpha c_{mr}$ , where  $c_{mr}$  is the modified phase speed obtained by summing the real part of (5) by the Shanks method). This curve is in very good agreement with Herbert's (1977) curve obtained by the FT method using  $N = 4$ . The points lying at the two extremities of the curves in figure 10 were cross-checked by sixteen terms in the Stuart-Laudau series. In the remaining range ten terms were found to be adequate. It was found that the distorted fundamental wave contributed the most to the  $u'$  fluctuation, and the content of higher harmonics was small in  $u'$ . This last result also agrees with Herbert's (1977) work, and the experimental work of Nishioka *et al.* (1975).

Of particular interest in the  $u'_m$  versus  $\beta$  curve is the point at  $\alpha = 1.12$  and  $R = 5000$ , for which we obtained a value of  $u'_m = 0.02155$ . Herbert obtained  $u'_m = 0.0216$  at the same point using the FT method with  $N = 4$  (this result is reported in Herbert (1980, p. 247)). Moreover, in the present work, the  $u'_m$  versus  $\beta$  curve spans the (poorly) convergent region of the series, as well as the divergent region of the series. Thus the corroboration of the numerical results by two independent analytical approaches, viz Herbert's FT method and the present method, speaks very significantly of the correctness of the results by both methods, for the two-dimensional problem.

In the experimental curve obtained by Nishioka *et al.*, for  $u'_m$  versus  $\beta$ , a dip is seen in the values of  $u'_m$  for  $\beta$  around 0.36. This dip is not found in either Herbert's (1977) results or in the present results. This is not really a cause for concern, since Nishioka *et al.* report that the dip is due to the advent of three-dimensional spot-like disturbances, and three-dimensional disturbances are outside the scope of the present investigation.

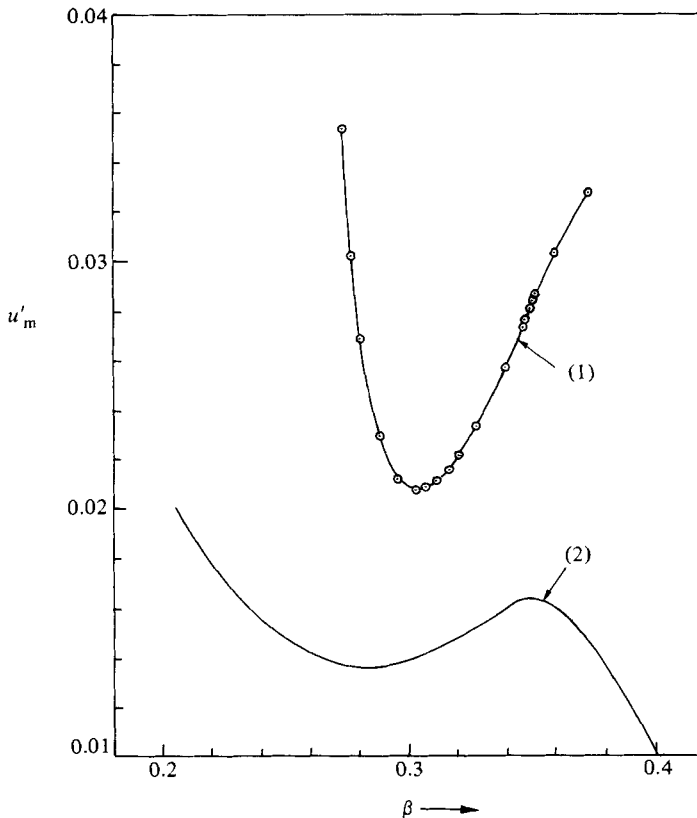


FIGURE 10. Plot of the maximum  $u'$  r.m.s. velocity  $u'_m$  versus the circular frequency  $\beta$ . Curve (1) is from the present work; curve (2) is from the experimental work of Nishioka *et al.* (1975). Curve (1) is virtually indistinguishable from that obtained by Herbert (1977) using the Fourier-truncation method with  $N = 4$ .

Incidentally, Zhou (1982) also obtains  $u'_m$  versus  $\beta$  curves that seem to be in better agreement with the experimental curves, although Zhou also considers two-dimensional disturbances. The formulation developed and used by Zhou is a very elegant one, and is midway between the FT method and the RP method. Thus one would expect that Zhou's calculations would tally with both these methods, because the results by the latter two methods are in very good agreement with each other. However, this is not found to be the case. For instance, the equilibrium amplitudes calculated by Zhou for the points ( $\alpha = 1.03$ ,  $R = 5500$ ), ( $\alpha = 1.04$ ,  $R = 5000$ ) and ( $\alpha = 1.05$ ,  $R = 4500$ ) are given respectively as 0.00378, 0.00635 and 0.00683. The results obtained in the present work, for the same set of points, are respectively 0.00413, 0.00821 and 0.01364. The point at  $\alpha = 1.04$  and  $R = 5000$  was also checked out by the  $V_p^{(m)}$  method as well, and the details are given in table 2. We thus note that Zhou's results are not in agreement with the present results or those by the FT method of Herbert (1977). A possible source of error in Zhou's results has already been suggested in §6.4. A look at Zhou's (1982) table 1 shows that the numerical transients in Zhou's series are not negligible. By looking at Zhou's equations (3.2) and (3.3) it seems that there is a need for the following. If the convergence rate is not rapid enough either in the various series in powers of  $a$ , or in those of powers of  $\epsilon$ , then such series should be correctly summed after evaluation of some more terms in the series, and in conjunction with a method like the Shanks method. Before this is done, it is perhaps too early to reach any conclusions regarding Zhou's results.

## 7. Comparison of the Watson method and the Reynolds & Potter method

In the Watson method the mean-motion equation (8) runs into difficulties for  $c_1 < 0$ , as mentioned earlier. Singularities appear whenever the following equality holds:

$$-2m\alpha c_1 R = \frac{1}{4}(2n+1)^2 \pi^2 \quad (c_1 < 0; \quad m = 1, 2, 3, \dots; \quad n = 0, 1, 2, 3, \dots). \quad (46)$$

This is illustrated in table 4. Actually, the smaller  $c_1$  becomes, the higher is the order of  $m$  at which the equality in (46) holds for the first time ( $n = 0$ ). Now, the order of  $m$  is the same as the order of the mean-motion distortion function  $F_m$ . Thus a blow up of  $F_m$  at a singularity results in the blow up of the corresponding value of  $K_m$ , though the  $K_n$  for  $n < m$  do not blow up. In table 4, entries (3), (5) and (7) correspond approximately to the blow up of  $K_3$ ,  $K_2$  and  $K_1$  respectively; i.e. with  $n = 0$  and  $m = 3, 2, 1$  respectively in (46). Also, entry (9) in table 4 corresponds approximately to the blow up of  $K_3$ , this time with  $n = 1$  and  $m = 3$  in (46).

The remaining entries in table 4 indicate some parity in the  $K_n$  coefficients up to  $n = 3$  with the RP method. These are the points where direct encounters with singularities have been avoided, although it is difficult to avoid diffused influence of singularities in the entire region of  $c_1 < 0$ , in view of the nature of (46).

The conclusion is that Watson's method should not be used in the region  $c_1 < 0$ , unless a suitable modification is made in the theory, to eliminate the problem of singularities. However, for  $c_1 \rightarrow 0$ , in the region  $c_1 < 0$ , the Stuart–Landau series in Watson's method is definitely of asymptotic validity, because the blow up occurs at increasingly higher orders in  $m$  with  $c_1 \rightarrow 0$ . The incentive to improve Watson's theory in the region  $c_1 < 0$  is provided by the fact that this problem is the 'true problem', which can give the behaviour of  $d|A|^2/dt$  for different levels of  $|A|$ . The 'false problem' of RP is valid only at the equilibrium state, and when equilibrium exists; although it can be used with impunity in the region  $c_1 < 0$ .

We next consider the supercritical region, i.e. with  $c_1 > 0$ . The results in table 5 again show some measure of parity in the  $K_n$  coefficients up to  $n = 3$ , for the Watson method and the RP method. However, for  $n > 3$ , the nature of  $K_n$  coefficients in the two methods are different. This is shown in table 6, for the point at  $\alpha = 0.826$  and  $R = 9000$ . In case of the RP method, the  $K_{ni}$  coefficients are of alternating signs, and for the Watson method the  $K_{ni}$  coefficients eventually have the same sign.

Since the  $K_{ni}$  coefficients are eventually of the same sign in Watson's method, the nearest singularity (corresponding to the radius of convergence) in the Stuart–Landau series lies on the positive  $|A|^2$  axis. Nevertheless, if the equilibrium amplitude is even slightly less than the radius of convergence, there is no doubt about its correctness, because for  $|A| < r_c$  the series has a unique sum, which can very reliably be obtained by the Shanks method. Table 6 shows that the radius of convergence in Watson's method is considerably smaller than that in the RP method. At the point considered, the former has  $r_c = 0.005$ , and the latter has  $r_c = 0.012$ . Further, the RP method has the nearest singularity on the negative  $|A|^2$  axis, which means that the nearest singularity poses no problem if the Shanks method is used to sum the series. The series in Watson's method, on the other hand, has closely packed singularities (see figure 11) for  $|A| > r_c$ . Thus Watson's method cannot be used for  $|A| > r_c$  in this region, even in conjunction with the Shanks method. It seems therefore that the RP method proves to be superior to the Watson method in the supercritical region, for the present problem of plane Poiseuille flow.

The results given in table 6 have been used to obtain  $S$  versus  $|A|$  curves (see (15))

Serial no.	R	Method	$c_r$	$c_i$ ( $\times 10^{-3}$ )	$K_{1r}$	$K_{1i}$	$K_{2r}$ ( $\times 10^6$ )	$K_{2i}$ ( $\times 10^6$ )	$K_{3r}$ ( $\times 10^{10}$ )	$K_{3i}$ ( $\times 10^{10}$ )
1	5774	W RP	0.2639	-0.002068	-140.4 -140.6	-28.85 -28.88	-25.98 8.585	-1.361 2.530	100.9 -0.6555	11.05 -0.2341
2	5750	W RP	0.2641	-0.04190	-134.2 -140.2	-28.05 -28.74	-60.47 8.523	-5.229 2.501	449.7 -0.6484	50.16 -0.2301
3	5730	W RP	0.2643	-0.07541	-126.2 -139.9	-27.08 -28.62	-168.2 8.471	-17.32 2.476	-5928 -0.6425	-664.0 -0.2267
4	5720	W RP	0.2644	-0.09228	-120.6 -139.8	-26.41 -28.56	-434.1 8.446	-47.15 2.464	-3259 -0.6396	-364.7 -0.2251
5	5710	W RP	0.2644	-0.1092	-113.2 -139.6	-25.54 -28.50	2042 8.420	230.7 2.452	7884 -0.6366	879.5 -0.2235
6	5700	W RP	0.2645	-0.1252	-103.0 -139.4	-24.35 -28.44	389.6 8.395	45.24 2.440	932.7 -0.6337	103.5 -0.2218
7	5650	W RP	0.2649	-0.2124	3037 -138.6	329.7 -28.15	-50119 8.268	-5637 2.381	1207380 -0.6192	135596 -0.2138
8	5500	W RP	0.2662	-0.4823	-185.0 -136.2	-33.12 -27.31	3.305 7.892	2.525 2.208	-9.549 -0.5771	-2.607 -0.1908
9	5400	W RP	0.2670	-0.6723	-176.1 -134.6	-32.09 -26.77	10.66 7.647	4.046 2.097	3482 -0.5502	1642 -0.1765
10	5200	W RP	0.2688	-1.078	-170.9 -131.4	-32.18 -25.74	-57.19 7.168	-31.25 1.888	-38.41 -0.4989	-21.33 -0.1501

TABLE 4. Results for the region  $c_1 < 0, \alpha = 1.02$ . W stands for Watson and RP for Reynolds & Potter



Serial no.	$R$	$\alpha$	Method	$c_r$	$c_1$ ( $\times 10^{-3}$ )	$K_{lr}$	$K_{li}$	$K_{sr}$ ( $\times 10^5$ )	$K_{si}$ ( $\times 10^5$ )	$K_{sr}$ ( $\times 10^{10}$ )	$K_{si}$ ( $\times 10^{10}$ )
1	5780	1.02	W	0.2639	0.006215	-141.4	-28.98	-22.18	-0.9351	81.12	8.832
			RP			-140.7	-28.91	8.600	2.537	-0.6573	-0.2351
2	6000	1.00	W	0.2598	0.3181	-146.5	-23.09	5.986	1.393	0.3615	-0.04530
			RP			-133.2	-21.42	7.673	1.693	-0.5526	-0.1390
3	6000	1.05	W	0.2656	0.1335	-173.6	-48.11	-2.571	3.502	27.46	1.919
			RP			-162.9	-47.00	11.87	5.227	-1.065	-0.6079
4	7000	1.05	W	0.2584	1.106	-201.7	-61.68	16.53	6.611	-2.298	-1.063
			RP			-182.7	-58.33	15.63	7.895	-1.605	-1.092
5	9000	0.830	W	0.2212	0.3771	-96.55	9.076	3.065	-1.202	-0.05630	0.1088
			RP			-85.32	10.54	2.630	-1.375	-0.08338	0.1520
6	9000	0.850	W	0.2239	1.342	-106.5	8.539	3.239	-0.9029	-0.1271	0.06330
			RP			-92.76	11.53	3.267	-1.398	-1.1259	0.1567
7	10000	1.05	W	0.2425	2.152	-259.5	-109.5	31.61	16.06	-6.787	-3.511
			RP			-247.9	-102.0	30.37	20.90	-4.204	-4.277

TABLE 5. Results for the region  $c_1 > 0$ ; W stands for Watson and RP for Reynolds & Potter

$n$	RP		Watson	
	$K_{n1}$	$A_n$	$K_{n1}$	$A_n$
1	$0.1022 \times 10^2$	0.003998	$0.9135 \times 10$	0.004229
2	$-0.1360 \times 10^6$	—	$-0.1142 \times 10^6$	—
3	$0.1505 \times 10^{10}$	0.004494	$0.1582 \times 10^{10}$	0.004704
4	$-0.1488 \times 10^{14}$	—	$0.4403 \times 10^{13}$	0.004687
5	$0.1347 \times 10^{18}$	0.004531	$0.2542 \times 10^{18}$	0.004667
6	$-0.1129 \times 10^{22}$	—	$0.1015 \times 10^{23}$	0.004651
7	$0.8823 \times 10^{25}$	0.004532	$0.3558 \times 10^{27}$	0.004639
8	$-0.6374 \times 10^{29}$	—	$0.1334 \times 10^{32}$	0.004631
9	$0.4096 \times 10^{33}$	0.004532	$0.5184 \times 10^{36}$	0.004624

TABLE 6.  $\alpha = 0.826$ ,  $R = 9000$ ,  $c = 0.2207 + 0.0001634i$ . Normalization of  $\psi_1$ :  $\psi_1(0) = 1$ , for both  $K_n$  as well as amplitudes. Equilibrium amplitude by Shanks method  $A_e = 0.004532$  (RP),  $A_e = 0.004604$  (Watson); radius of convergence  $r_c = 0.012$  (RP),  $r_c = 0.005$  (Watson). Note:  $A_n$  is the equilibrium amplitude based on the direct sum for  $S$ , (15), up to the  $K_{n1}$  term.

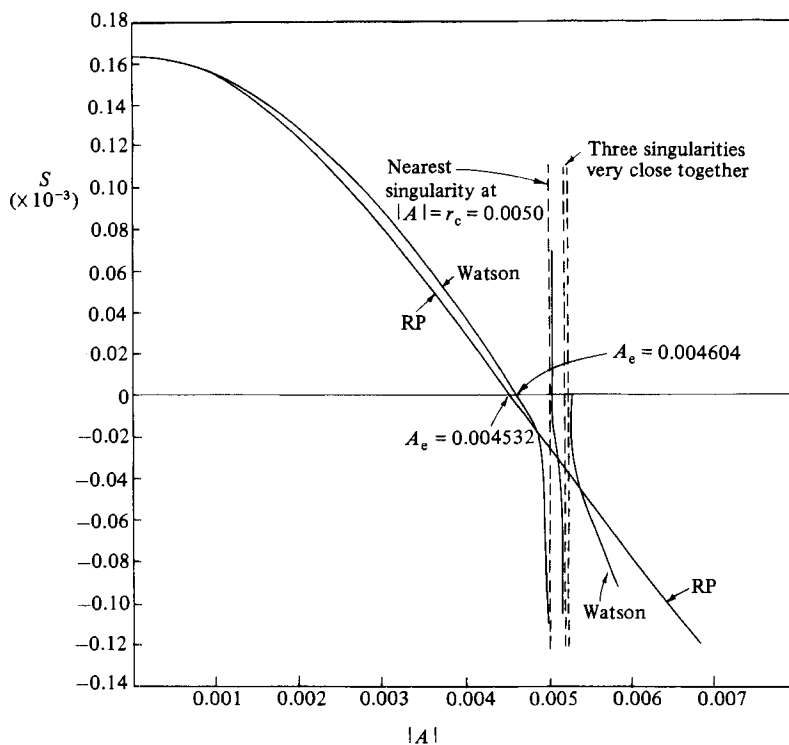


FIGURE 11.  $S$  versus  $|A|$  curves in the supercritical ( $c_1 > 0$ ) region at  $\alpha = 0.826$ ,  $R = 9000$ . Sum of the series  $S$  (15) was obtained by the Shanks method. Curve for Watson's method shows closely packed singularities, beyond the nearest singularity. See also table 6.

for both the methods, using the Shanks method of series summation. These curves are shown in figure 11. The result is revealing. First, the pole in Watson's method is clearly seen to appear for  $|A| = 0.005$ . Secondly, the equilibrium amplitudes obtained by both the methods are almost the same. The RP method gives  $A_e = 0.004532$ , and the Watson method gives  $A_e = 0.004604$ , both of which were calculated by the Shanks method. The good agreement obtained in  $A_e$  adds credence to the numerical work, and to the correctness of either formulation.

Figure 11 also reveals a very interesting feature. It is seen that the shapes of the  $S$  versus  $|A|$  curves, by both the methods, are more or less the same for  $|A| \leq A_e$ . Thus it appears that the 'false problem' is perhaps not that 'false' after all. This result can possibly be capitalized upon, in the subcritical region, where the 'true problem' of Watson cannot be used, although at this stage this result cannot be verified for the subcritical region. Perhaps some collateral verification of this point will be obtained by considering the stability problem for boundary-layer flow past a flat plate. For this problem, the mean-motion equation in Watson's method is not subjected to singularities, provided that non-parallel effects are ignored. Nevertheless, the equivalence of the results by the two methods, for  $|A| \leq A_e$  in the supercritical region in the present problem, is a very encouraging feature in favour of the RP method.

As a final word of comment, it must be emphasized that all of the various conclusions that have been drawn so far, from the results for the Watson method and the RP method in the present problem of plane Poiseuille flow, should not automatically be assumed to be true for other cases of parallel or near-parallel flows. These other problems need to be worked out separately, and everything will depend on the nature of the respective  $K_n$  coefficients obtained in these problems.

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